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Action principles for the elastic solid and the perfect fluid in general relativity

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Abstract. Action principles using the Eulerian description are proposed for the elastic solid and perfect fluid in general relativity. By taking into account the constraints on the independent variables appearing in the Lagrangian density, it is shown that these action principles are equivalent to those using the Lagrangian description, which have been given previously.

1. Introduction

Action principles for elastic solids and perfect fluids in general relativity have been given by DeWitt (1962) and Taub (1954). These authors make use of the Lagrangian description, as opposed to the Eulerian description. The action principle for the electromagnetic field is naturally given in the Eulerian description, and therefore the interaction of the electromagnetic field with matter is more easily described if an action principle for matter is available in this description. Such action principles for both the elastic solid and the perfect fluid are given here.

The Lagrangian densities in both cases are constructed to conform to the following formalism (Pauli 1958). Let R be the curvature scalar of the spacetime manifold, and $\mathscr{R}$ the associated scalar density, so that $\mathscr{R} = (g)^{1/2}R$, where $g = -\det g_{ik}$ and the g_{ik} ($i, k, = 1, 2, 3, 4$) are the components of the fundamental tensor; the associated fundamental quadratic form $\Phi = g_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k$ is assumed to have signature $+2$, in accord with Synge's notation (Synge 1960). Then if $\mathscr{L}$ is the Lagrangian density associated with the presence of matter, the gravitational field equations follow from the action principle

$$\delta A = 0 \tag{1}$$

with

$$A = \int \left(-\frac{\mathscr{R}}{2\kappa} + \mathscr{L}\right) \mathrm{d}^4x. \tag{2}$$

In (2), κ is the gravitational constant, and the integral is over a fixed region of the spacetime manifold. The action principle (1) holds for those variations of the g_{ik}, and the other variables occurring in $\mathscr{L}$, that vanish on the boundary of the region of integration.

The gravitational field equations obtained from (1) are

$$\mathscr{G}^{ik} + \kappa\mathscr{T}^{ik} = 0$$

where $\mathscr{G}^{ik}$ is the tensor density associated with the Einstein tensor G^{ik} and $\mathscr{T}^{ik}$ is defined by

$$\mathscr{T}^{ik} = 2\frac{\delta\mathscr{L}}{\delta g_{ik}}. \tag{3}$$

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$\delta\mathscr{L}/\delta g_{ik}$ denotes the Lagrange derivatives of $\mathscr{L}$ with respect to the field g_{ik}. The matter field equations are not obtained directly from (1), however, for in the Eulerian description, the variables describing the motion of a material continuum cannot be varied independently, but must be subjected to certain constraints. The constraints necessary in the nonrelativistic case have been discussed recently by Seliger and Whitham (1968), and in the special relativistic case by Penfield and Haus (1967). The constraints used here are

$$g_{ik}w^i w^k + 1 = 0 \tag{4}$$

$$w^i \partial_i X^K = 0 \tag{5}$$

$$\partial_i(\rho w^i) = 0. \tag{6}$$

The first constraint, (4), represents the condition that the world velocity vector w^i of the motion should be a unit timelike vector. (5) is the relativistic generalization of Lin's constraint (Seliger and Whitham, 1968), the $X^K(K = 1, 2, 3)$ being a set of three scalar fields labelling the material points of the continuum. (6) represents the conservation of mass, ρ being the mass density, which transforms as a scalar density.

The matter field equations can now be derived from (1) using the method of Lagrange multipliers to take account of the constraints (4)–(6). Let α, β_K, and γ be the Lagrange multipliers then as far as the derivation of the matter field equations is concerned, (1) is equivalent to

$$\delta A = 0 \tag{7}$$

where

$$A' = \int \mathscr{L}' \, d^4x$$

and

$$\mathscr{L}' = \mathscr{L} + \alpha(g_{ik}w^i w^k + 1) + \beta_K w^i \partial_i X^K + \gamma \partial_i(\rho w^i).$$

The Lagrange multipliers are to be regarded as variables on an equal footing with the g_{ik} and the variables used to describe the material continuum. Their variation in (7) leads simply to the equations of constraint (4)–(6).

2. The elastic solid

The elastic solid is assumed here to be characterized by the following functional form for $\mathscr{L}$,

$$\mathscr{L} = -\rho(1+\Sigma)(-g_{ik}w^i w^k)^{1/2} \tag{8}$$

where Σ, the internal energy density per unit mass, is a function of the g_{ik} and the $\partial_i X^K$ only. For $\mathscr{L}$ to be a scalar density Σ must be an absolute scalar, and must therefore be expressible in terms of scalars formed from the g_{ik} and the $\partial_i X^K$. To ensure that the theory reduces to classical elasticity theory in the nonrelativistic approximation, Σ is assumed to depend only on the world scalars $\overset{-1}{C}{}^{KL}$ defined by

$$\overset{-1}{C}{}^{KL} = g^{kl}\partial_k X^K \partial_l X^L.$$

The $\overset{-1}{C}{}^{KL}$ represent the components of the inverse of Green's deformation tensor in the instantaneous rest frame of the material at a given spacetime point (Toupin 1957, Grot and Eringen 1966). The derivatives of Σ with respect to g_{ik} and $\partial_i X^K$ will be useful later. They are

$$\frac{\partial \Sigma}{\partial g_{ik}} = \frac{\partial \Sigma}{\partial \overset{-1}{C}{}^{MN}} \partial_m X^M \partial_n X^N g^{mi} g^{nk}$$

and

$$\frac{\partial \Sigma}{\partial(\partial_i X^K)} = 2 \frac{\partial \Sigma}{\partial \overset{-1}{C}{}^{KN}} \partial_n X^N g^{ni}.$$

The tensor density $\mathscr{T}^{ik}$ associated with the Lagrangian density (8) is

$$\begin{aligned} \mathscr{T}^{ik} &= 2\frac{\delta \mathscr{L}}{\delta g_{ik}} = 2\frac{\partial \mathscr{L}}{\partial g_{ik}} \\ &= \rho(1+\Sigma) w^i w^k - t^{ik} \end{aligned} \tag{9}$$

where the constraint (4) has been used after the differentiation has been carried out. The tensor density t^{ik} is the relativistic generalization of the Cauchy stress tensor defined by Toupin (1960) and is defined by

$$\begin{aligned} t^{ik} &= 2\rho \frac{\partial \Sigma}{\partial g_{ik}} \\ &= -2\rho \frac{\partial \Sigma}{\partial \overset{-1}{C}{}^{MN}} \partial_m X^M \partial_n X^N g^{mi} g^{nk}. \end{aligned}$$

The field equations that follow from (7) with the special form (8) for $\mathscr{L}$ are

$$\rho(1+\Sigma) w_i + 2\alpha w_i + \beta_K \partial_i X^K - \rho \partial_i \gamma = 0 \tag{10}$$

$$1+\Sigma + w^i \partial_i \gamma = 0 \tag{11}$$

$$\partial_i \rho \frac{\partial \Sigma}{\partial(\partial_i X^K)} - \beta_K w^i = 0 \tag{12}$$

together with the constraint equations, which have been used to simplify the above. Multiplying (10) by w^i and using (11) leads at once to the result that $\alpha = 0$, so that (10) can be replaced by

$$\rho(1+\Sigma) w_i + \beta_K \partial_i X^K - \rho \partial_i \gamma = 0. \tag{13}$$

Multiplying (13) by w^k and differentiating with respect to x^k leads, after using the constraint equations, to

$$\partial_k \mathscr{T}^k_{\ i} - \rho \frac{\partial \Sigma}{\partial(\partial_k X^K)} \partial^2_{ik} X^K + \beta_K w^k \partial^2_{ik} X^K - \partial_i(\rho w^k \partial_k \gamma) + \partial_i(\rho w^k) \partial_k \gamma = 0. \tag{14}$$

Substituting in (14) for $\rho w^k \partial_k \gamma$ and $\partial_k \gamma$, using (11) and (13), leads, again after some manipulation and the use of constraints, to

$$\partial_k \mathscr{T}^k_{\ i} - \rho \frac{\partial \Sigma}{\partial(\partial_k X^K)} \partial^2_{ik} X^K + \rho \partial_i \Sigma + \rho(1+\Sigma) w_k \partial_i w^k = 0. \tag{15}$$

Now

$$\partial_i \Sigma = \frac{\partial \Sigma}{\partial(\partial_k X^K)} \partial^2_{ik} X^K + \frac{\partial \Sigma}{\partial g_{kl}} \partial_i g_{kl}$$

and therefore (15) reduces to

$$\partial_k \mathscr{T}^k_i + \rho \frac{\partial \Sigma}{\partial g_{kl}} \partial_i g_{kl} + \rho(1+\Sigma) w_k \partial_i w^k = 0. \tag{16}$$

Also

$$\partial_i(g_{kl} w^k w^l) = 0$$

therefore

$$w_k \partial_i w^k = -\tfrac{1}{2} w^k w^l \partial_i g_{kl}.$$

Thus (16) becomes

$$\partial_k \mathscr{T}^k_i - \tfrac{1}{2} \mathscr{T}^{kl} \partial_i g_{kl} = 0$$

therefore

$$\nabla_k \mathscr{T}^k_i = 0 \tag{17}$$

where ∇_k denotes covariant differentiation defined with respect to the Christoffel symbols formed from the g_{ik}.

Equation (17), together with the equations of constraint (4)–(6), constitutes the system of field equations for the elastic solid obtained in the more usual Lagrangian description (DeWitt 1962). The above argument shows, therefore, that every solution of the field equations following from the action principle (7), with the special form (8) for $\mathscr{L}$ is also a solution of the field equations obtained from the usual action principle. To complete the proof of the equivalence of the two systems of the field equations, it remains to show that any solution of (17) and the constraints (4)–(6) is also a solution of the equations (10)–(12). That is, given ρ, w^i, X^K and g_{ik} as functions of the coordinates satisfying (17) and (4)–(6), it must be shown that there exist functions α, β_K, and γ such that the two sets of functions taken together satisfy (10)–(12).

This can be done as follows. First put $\alpha = 0$ and let γ be any solution of (11), regarding (11) as a first-order partial differential equation for γ. Then the functions β_K are uniquely determined by (1). Explicitly, define 12 quantities x^i_K by the equations

$$w_i x^i_K = 0 \tag{18}$$

$$x^i_L \partial_i X^K = \delta^K_L \tag{19}$$

(18) and (19) uniquely define the x^i_K, for the four vectors w_i, $\partial_i X^K$ are linearly independent (Toupin, 1957). Using (18) and (19), it follows from (10) that

$$\beta_K = \rho x^i_K \partial_i \gamma. \tag{20}$$

Equations (10) and (11) are now automatically satisfied; it remains to show that (12) is satisfied. Using (20) the lhs of (12) becomes, apart from sign,

$$\mu \partial_i (t^i_k x^k_K + \rho x^k_K \partial_k \gamma w^i). \tag{21}$$

Useful formulas in the reduction of (21) are

$$\partial_i x^k_K = w^k x^l_K \partial_i w_l - x^k_L x^l_K \partial_{il}{}^2 X^L,$$

$$\partial_i X_K x^k_K = g^k_i + w_i w^k.$$

Coupling these results with (11) and (17), it is a matter of straightforward, if tedious, calculation to show that (21) vanishes, and hence that (12) is satisfied by the above choice of α, β_K, and γ. Thus the equivalence of the systems of equations following from the action principle proposed here and the more usual one is established.

3. The perfect fluid

The scalar density $\mathscr{L}$ for a perfect fluid has the same form (8) as that for an elastic solid, but the internal energy Σ is now assumed to depend only on the mass density ρ_0 as measured in the instantaneous rest frame of the material. The relation of this density to ρ can be obtained from a consideration of the invariant volume d_3v in the three-space orthogonal to w^i at the spacetime point x^i. Let λ^i_α ($\alpha = 1, 2, 3$) be an orthonormal triad in this 3-space, and consider the volume element spanned by the infinitesimal vectors $d\xi^1\lambda^i_{(1)}$, $d\xi^2\lambda^i_{(2)}$, $d\xi^3\lambda^i_{(3)}$. The invariant volume element d_3v is just $d\xi^1\, d\xi^2\, d\xi^3$, while the corresponding element of extension in the 4-space is given by the antisymmetric tensor

$$dV^{ikm} = 3!\,\lambda^{[i}_{(1)}\lambda^k_{(2)}\lambda^{m]}_{(3)}\, d_3v.$$

The dual vector density dV_n is

$$dV_n = \epsilon_{ikmn}\lambda^i_{(1)}\lambda^k_{(2)}\lambda^m_{(3)}\, d_3v$$

where ϵ_{ikmn} is the permutation symbol. The vector density

$$\gamma_n = \epsilon_{ikmn}\lambda^i_{(1)}\lambda^k_{(2)}\lambda^m_{(3)}$$

is orthogonal to each of the $\lambda^i_{(\alpha)}$ by inspection, and must therefore be proportional to $w_n = g_{nm}w^m$ since w^i is also orthogonal to all the $\lambda^i_{(\alpha)}$. Therefore,

$$\gamma_n = \gamma g_{nm}w^m$$

and

$$g^{ln}\gamma_l\gamma_n = \gamma^2 g_{ik}w^i w^k.$$

But by direct calculation, it may be shown that

$$g^{ln}\gamma_l\gamma_n = -g^{-1}$$

so that

$$\gamma = g^{-1/2}(-g_{ik}w^i w^k)^{-1/2}.$$

ρ_0 may be defined by the relation

$$\begin{aligned}\rho_0\, d_3v &= -\rho w^n\, dV_n\\ &= -\rho w^n \gamma_n\, d_3v\\ &= -\rho w^n \gamma g_{nm} w^m\, d_3v\\ &= +\rho g^{-1/2}(-g_{ik}w^i w^k)^{+1/2}\, d_3v.\end{aligned}$$

Hence

$$\rho_0 = \rho g^{-1/2}(-g_{ik}w^i w^k)^{1/2}.$$

Notice that the factor in brackets equals unity when the constraint (4) is taken into account, but it is essential that this simplification is not made until after the required variations have been carried out, since it is the particular functional dependence of $\mathscr{L}$ on its independent variables, rather than its actual value, which is important. With this proviso, the following derivatives may be noted:

$$\frac{\partial \Sigma}{\partial \rho} = \frac{\partial \Sigma}{\partial \rho_0} g^{-1/2}$$

$$\frac{\partial \Sigma}{\partial w^i} = -\rho_0 \frac{\partial \Sigma}{\partial \rho_0} w_i$$

$$\frac{\partial \Sigma}{\partial g_{ik}} = -\tfrac{1}{2}\rho_0 \frac{\partial \Sigma}{\partial \rho_0}(g^{ik} + w^i w^k).$$

Using the last of these results, the tensor density $\mathscr{T}^{ik}$ for the perfect fluid may be calculated to be

$$\mathscr{T}^{ik} = \rho(1+\Sigma)w^i w^k - t^{ik}$$

where

$$t^{ik} = 2\rho \frac{\partial \Sigma}{\partial g_{ik}} = -\rho\rho_0 \frac{\partial \Sigma}{\partial \rho_0}(g^{ik} + w^i w^k).$$

The scalar pressure p may be defined by

$$p = \rho_0^2 \frac{\partial \Sigma}{\partial \rho_0}$$

and then

$$\mathscr{T}^{ik} = (g)^{1/2} T^{ik}$$

where

$$T^{ik} = \rho_0\left(1 + \Sigma + \frac{p}{\rho_0}\right) w^i w^k + p g^{ik}.$$

The field equations corresponding to the action principle (7) are

$$\rho(1+\Sigma)w_i + \rho\rho_0 \frac{\partial \Sigma}{\partial \rho_0} w_i + 2\alpha w_i + \beta_K \partial_i X^K - \rho \partial_i \gamma = 0 \tag{22}$$

$$1 + \Sigma + \rho_0 \frac{\partial \Sigma}{\partial \rho_0} + w^i \partial_i \gamma = 0 \tag{23}$$

$$\partial_i(\beta_K w^i) = 0 \tag{24}$$

which once again must be augmented with the equations of constraint (4)–(6). In exactly the same way as for the elastic solid, it may be shown that elimination of the Lagrange multipliers leads to the equation

$$\nabla_k \mathscr{T}_i{}^k = 0. \tag{25}$$

As in the case of the elastic solid, (25) and the constraint equations (4)–(6) make up the usual field equations, and so the elimination of the Lagrange multipliers shows that every solution of (22)–(24) and the constraints is also a solution of the usual field equations. That every solution of the latter system of equations is also a solution of (22)–(24) again follows from a tedious calculation. α may be put equal to zero, and γ may be taken to be any solution of (23), whereupon β_K is uniquely determined by (22). The vanishing of the left hand side of (24) then follows from (25), and hence, just as in the case of the elastic solid, the equivalence of the usual system of field equations with that following from the action principle (7), is established.

4. Conclusions

Action principles have been proposed, in the Eulerian description, for two special cases in continuum mechanics, the elastic solid and the perfect fluid. It has been shown, by taking into account the constraints on the independent variables in the Lagrangian density, that the action principles used here are equivalent to those usually used, which employ the Lagrangian description. A later paper discussing electromechanical interactions will illustrate the advantages of the Eulerian description in this respect.

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